Inference from Small Samples

If sample size**, n, is smaller than 30**, the test statistic of sample mean is a **Student's t-distribution**, with ***n*−1 degree of freedom**.

Small Sample Inference for a Population Mean μ:

Test $H_0 : \mu = \mu_0$ versus H_1 : one or two tailed using the test statistic $t = \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}}$ using p-values or a rejection region based on a t-distribution with $df = n = 1$.

<u>Example:</u>

A sprinkler system is designed so that the average time for the sprinklers to activate after being turned on is no more than 15 seconds. A test of 5 systems gave the following times:

17, 31, 12, 17, 13, 25

Is the system working as specified? Test using α = .05.

$H_0 : \mu = 15$ (Working as specified)

$H_a : \mu > 15$ (Not working as specified)

Data: 17, 31, 12, 17, 13, 25

First, calculate the sample mean and standard deviation.

$$\bar{x} = \frac{\sum x_i}{n} = \frac{115}{6} = 19.167$$

$$s = \sqrt{\frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1}} = \sqrt{\frac{2477 - \frac{115^2}{6}}{5}} = 7.387$$

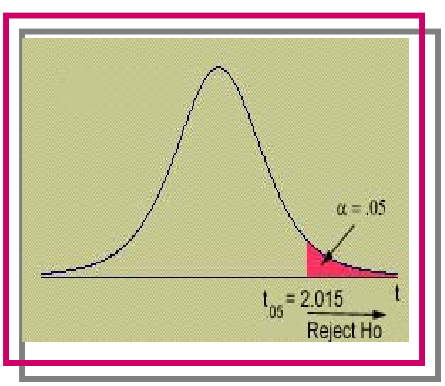
Calculate the test statistic and find the rejection region for α = .05.

Test statistic:

$$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}} = \frac{19.167 - 15}{7.387 / \sqrt{6}} = 1.38$$

Degrees of freedom:

$$df = n - 1 = 6 - 1 = 5$$



Rejection Region: Reject H_0 if $t >$ 2.015.

If the test statistic falls in the rejection region, its *p*-value will be less than α = .05.

Conclusion: For our example, *t* = 1.38 does not fall in the rejection region and H_0 is not rejected. There is insufficient evidence to indicate that the average activation time is greater than 15.

Difference between two means under small sample condition:

Case I: ***Paired Samples***:

Take the difference of each pair and treat the differences as one population case.

Case II: ***Independent*** **Samples with** ***Equal*** **Population Variances**:

Put two sample variances together to get a common variance. The test statistic will be

$$t \approx \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad \text{where} \quad s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

It is a t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Case III: ***Independent*** **Samples with** ***unequal*** **population variances:**

The test statistic is $$t \approx \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

It has a t-distribution with degrees of freedom as

$$df \approx \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

How can you tell if the equal variance assumption reasonable?

Rule of Thumb:

If the ratio, $\frac{\text{larger } s^2}{\text{smaller } s^2} \le 3$,

the equal variance assumption is reasonable.

If the ratio, $\frac{\text{larger } s^2}{\text{smaller } s^2} > 3$,

use an alternative test statistic.

Example:

Car	1	2	3	4	5
Type A	10.6	9.8	12.3	9.7	8.8
Type B	10.2	9.4	11.8	9.1	8.3

•One Type A and one Type B tire are randomly assigned to each of the rear wheels of five cars. Compare the average tire wear for types A and B using a test of hypothesis.

$$\text{H}_0 : \mu_1 - \mu_2 = 0$$
$$\text{H}_a : \mu_1 - \mu_2 \neq 0$$

•But the samples are not independent. The pairs of responses are linked because measurements are taken on the same car.

To test $\text{H}_0 : \mu_1 - \mu_2 = 0$ we test $\text{H}_0 : \mu_d = 0$, using the test statistic $t = \dfrac{\bar{d} - 0}{s_d / \sqrt{n}}$

where n = number of pairs, $\bar{d}$ and s_d are the mean and standard deviation of the *differences*, d_i.
Use the p-value or a rejection region based on a t-distribution with $df = n-1$.

Car	1	2	3	4	5
Type A	10.6	9.8	12.3	9.7	8.8
Type B	10.2	9.4	11.8	9.1	8.3
Difference	.4	.4	.5	.6	.5

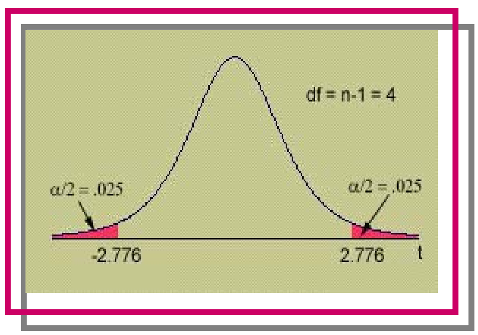
$$\text{H}_0 : \mu_1 - \mu_2 = 0$$
$$\text{H}_a : \mu_1 - \mu_2 \neq 0$$

Test statistic:

$$t = \frac{\bar{d} - 0}{s_d / \sqrt{n}} = \frac{.48 - 0}{.0837 / \sqrt{5}} = 12.8$$

Calculate $\bar{d} = \dfrac{\sum d_i}{n} = .48$

$$s_d = \sqrt{\frac{\sum d_i^2 - \frac{(\sum d_i)^2}{n}}{n-1}} = .0837$$



Rejection region: Reject H_0 if $t > 2.776$ or $t < -2.776$.
Conclusion: Since $t = 12.8$, H_0 is rejected. There is a difference in the average tire wear for the two types of tires.

Example:

Two training procedures are compared by measuring the time that it takes trainees to assemble a device. A different group of trainees are taught using each method. Is there a difference in the two methods? Use $\alpha = .01$.

Time to Assemble	**Method 1**	**Method 2**
Sample size	10	12
Sample mean	35	31
Sample Std Dev	4.9	4.5

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_a: \mu_1 - \mu_2 \neq 0$$

Test statistic:

$$t = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

•Solve this problem by approximating the p-value using t-Table.

Calculate:

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{9(4.9^2) + 11(4.5^2)}{20} = 21.942$$

Test statistic:

$$t = \frac{35 - 31}{\sqrt{21.942\left(\frac{1}{10} + \frac{1}{12}\right)}}$$

$$= 1.99$$

$$p\text{-value}: P(t > 1.99) + P(t < -1.99)$$

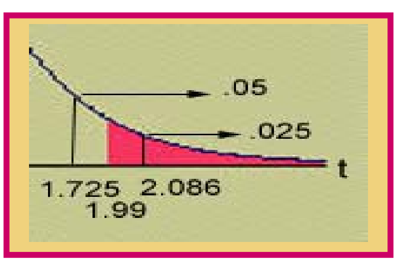
$$P(t > 1.99) = \frac{1}{2}(p\text{-value})$$

$$df = n_1 + n_2 - 2 = 10 + 12 - 2 = 20$$

$.025 < ½(p\text{-value}) < .05$

$.05 < p\text{-value} < .10$

Since the p-value is greater than $\alpha = .01$, H_0 is not rejected. There is insufficient evidence to indicate a difference in the population means.



df	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	df
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20